

The $su(N)$ XX model

Z. Maassarani and P. Mathieu ¹

Département de Physique, Université Laval, Québec, Canada G1K 7P4

Abstract

The natural $su(N)$ generalization of the XX model is introduced and analyzed. It is defined in terms of the characterizing properties of the usual XX model: the existence of two infinite sequences of mutually commuting conservation laws and the existence of two infinite sequences of mastersymmetries. The integrability of these models, which cannot be obtained in a degenerate limit of the $su(N)$ -XXZ model, is established in two ways: by exhibiting their R matrix and from a direct construction of the commuting conservation laws. We then diagonalize the conserved laws by the method of the algebraic Bethe Ansatz. The resulting spectrum is trivial in a certain sense; this provides another indication that the XX $su(N)$ model is the natural generalization of the $su(2)$ model. The application of these models to the construction of an integrable ladder, that is, an $su(N)$ version of the Hubbard model, is mentioned.

¹ Work supported by NSERC (Canada) and FCAR (Québec).

1. Introduction

Much efforts have been devoted in recent years to the study of the XXZ model and its Lie algebraic generalizations. These models have displayed a very rich mathematical structure and in particular an underlying quantum group symmetry [1]. In the $su(2)$ case, the XX model is obtained from the XXZ model

$$H^{\text{XXZ}} = \sum_i [\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z] \quad (1.1)$$

in the limit $\Delta = 0$. Its integrability is a somewhat trivial fact since the model is equivalent to a free-fermion model. One of the practical uses of the XX model is rooted in its role as a building block for the Hubbard model defined as [2]:

$$H^{\text{Hub}} = \sum_i [\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \tau_i^x \tau_{i+1}^x + \tau_i^y \tau_{i+1}^y + U \sigma_i^z \tau_i^z] \quad (1.2)$$

where the σ 's and the τ 's are two independent sets of Pauli matrices. In the present work, we consider the integrable $su(N)$ extension of the XX model. In contrast to the $N = 2$ case, the $N > 2$ models cannot be obtained as a limiting case of the one-parameter $su(N)$ R matrix related to $\mathcal{U}_q(sl(N))$ [1], nor from any known multiparameter deformation of the $su(N)$ XXX model (see e.g., [3]). Moreover, one may suspect that by simply dropping, from the isotropic XXX version, the contributions of the generators associated with the Cartan subalgebra will not lead to an integrable model. Hence, the first step amounts to a proper definition of the XX model through characterizing properties that are expected to be maintained in the generalized version. This is the subject of section 2. We then present the natural $su(3)$ extension of the $su(2)$ XX model and analyze it in some details. In particular, we display its R matrix and the explicit expression of all its conservation laws. These results are then generalized to the $su(N)$ case. We then diagonalize the transfer matrix by the method of the algebraic Bethe Ansatz. The resulting spectrum is trivial in a certain sense; this provides another indication that the XX $su(N)$ model is the natural generalization of the $su(2)$ model. The choice of characterizing properties of the $su(N)$ XX spin chain is further supported by a construction of the $su(N)$ extension of the Hubbard model, whose detailed study is left to a subsequent paper. All our results pertain to infinite or periodic chains.

2. The $su(2)$ XX model

To formulate the problem of finding the integrable generalization of the XX model in a well-defined way, we need some general characterization of the latter.

At first we recall that the XX model has two infinite sequences of conservation laws. If we denote by $H_n(\Delta)$ the unique XXZ conservation law whose leading term describes the interaction of n contiguous spins, then one sequence is simply $H_n^{(+)} = H_n(0)$ (the defining hamiltonian being $H_2^{(+)}$). There is a second independent sequence $\{H_n^{(-)}\}$ and all the members of the whole set $\{H_n^{(\pm)}\}$ commute among themselves. The explicit form of these laws is [4, 5, 6]:

$$\begin{aligned} H_n^{(+)} &= \mathcal{H}_n^{xx} + \mathcal{H}_n^{yy} & n \text{ even,} \\ &= \mathcal{H}_n^{xy} - \mathcal{H}_n^{yx} & n \text{ odd,} \\ H_n^{(-)} &= \mathcal{H}_n^{xy} - \mathcal{H}_n^{yx} & n \text{ even,} \\ &= \mathcal{H}_n^{xx} + \mathcal{H}_n^{yy} & n \text{ odd,} \end{aligned} \tag{2.1}$$

with

$$\mathcal{H}_n^{ab} = \sum_j h_{n,j}^{ab}, \quad h_{n,j}^{ab} = \sigma_j^a \sigma_{j+1}^z \cdots \sigma_{j+n-2}^z \sigma_{j+n-1}^b, \tag{2.2}$$

and $n \geq 2$. To the above set of conserved charges, we can obviously add

$$H_1^{(-)} = -2 \sum_j \sigma_j^z \tag{2.3}$$

(the factor -2 is introduced for later convenience.)

The standard XXZ R matrix does not explain the presence of two infinite sequences of conservation laws. However, there exists a one-parameter deformation of this R matrix [7] that leads to a two-parameter hamiltonian:

$$H = \sum [\cos \delta (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) - \sin \delta (\sigma_i^x \sigma_{i+1}^y - \sigma_i^y \sigma_{i+1}^x) + \cosh \gamma (\sigma_i^z \sigma_{i+1}^z)] \tag{2.4}$$

This reduces to the usual XXZ hamiltonian when $\delta = 0$. However with $\gamma = i\pi/2$, we obtain a linear combination of $H_2^{(+)}$ and $H_2^{(-)}$ and these two parts commute as already mentioned.

The property of having two infinite sequences of conservation laws does not provide a unique characterization of the model since it is also shared by the more general XYh model, the anisotropic version of the XX model in a magnetic field.

Another characteristic property of the XX model is the existence of an infinite number of mastersymmetries $B_n^{(\pm)}$ whose commutation with conservation laws generate conservation laws (this is the definition of a mastersymmetry). $B_n^{(\pm)}$ is the first moment of the charge $H_n^{(\pm)}$. For instance

$$B_2^{(+)} = \sum_j [j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y)] \quad (2.5)$$

and similarly, the lowest order mastersymmetry is simply $B_1^{(-)} = -2 \sum j \sigma_j^z$. The precise expression for the commutator of the mastersymmetries with the conservation laws is

$$[B_k^{(\epsilon_1)}, H_n^{(\epsilon_2)}] = 2i(-1)^\phi (n-1) [H_{n+k-1}^{(\epsilon_1 \epsilon_2)} + (-1)^\xi H_{|n-k|+1}^{(\epsilon_1 \epsilon_2)}] \quad (2.6)$$

with $\epsilon_i = \pm$; ϕ and ξ are functions of n, k, ϵ_i .² This formula holds for all positive values of n and k with the understanding that $H_1^{(+)} = 0$; in particular, it covers the case $k = 1$ which justifies the previous choice of normalization for $H_1^{(-)}$. It is thus clear that any conservation law, even $H_1^{(-)}$, can be obtained from $H_2^{(+)}$ by the application of a suitable mastersymmetry. Notice that the commutator of two mastersymmetries is also a mastersymmetry:

$$\begin{aligned} [B_k^{(\epsilon_1)}, B_n^{(\epsilon_2)}] = 2i(-1)^\phi \left\{ (n-k) B_{n+k-1}^{(\epsilon_1 \epsilon_2)} \right. \\ \left. + (-1)^\xi [(n+k-2) B_{|n-k|+1}^{(\epsilon_1 \epsilon_2)} - (\min(k, n) - 1)^2 H_{|n-k|+1}^{(\epsilon_1 \epsilon_2)}] \right\} \end{aligned} \quad (2.7)$$

As a side remark, we stress that (2.6) together with $[H_n^{(\pm)}, H_2^{(+)}] = 0$ imply the commutation of all the charges among themselves: $[H_n^{(\epsilon_1)}, H_m^{(\epsilon_2)}] = 0$. This is a consequence of the Jacobi identity with a recursion on n (with $m > n$).

Again, it is known that for the XYh model, there is also an infinite number of mastersymmetries [8, 5]. However, only half of the XX mastersymmetries ($B_2^{(+)}, B_3^{(-)}, B_4^{(-)}, \dots$)

² The explicit expression of the phases is

$$\phi = \frac{1}{4} [(-1)^k + \epsilon_1] [(-1)^n + \epsilon_2]$$

and

$$\xi = \begin{cases} \epsilon_1 (-1)^k & \text{for } k \leq n \\ \frac{1}{4} [(-1)^k + \epsilon_1] [(-1)^n - \epsilon_2] + 1 & \text{for } k > n \end{cases}$$

survive the anisotropic deformation³ (in particular, the sets $\{H_n^{(+)}\}$ and $\{H_n^{(-)}\}$ are not related by the action of a mastersymmetry). Hence, it appears that requiring two infinite sequences of mastersymmetries provides a unique characterization of the XX model. Most likely, this boils down to the simpler criterion of the existence of the two mastersymmetries $B_1^{(-)}$ and $B_2^{(+)}$.

3. The $su(3)$ XX model

3.1. Definition of the model

The standard form of the $su(3)$ XXZ model defined from the $\mathcal{U}_q(sl(3))$ R matrix, written in terms of the Gell-Mann matrices,⁴ reads

$$H = \sum_i \left[\sum_{a \neq 3,8} \lambda_i^a \lambda_{i+1}^a + \cosh \gamma (\lambda_i^3 \lambda_{i+1}^3 + \lambda_i^8 \lambda_{i+1}^8) - \frac{1}{\sqrt{3}} \sinh \gamma (\lambda_i^3 \lambda_{i+1}^8 - \lambda_i^8 \lambda_{i+1}^3) \right] \quad (3.1)$$

There is no value of the parameter γ that decouples the Cartan subalgebra generators. Therefore the suitable form of the $su(3)$ XX hamiltonian must be guessed. As a first selecting condition we require the model to have an extra conservation law that couples at most two spins (i.e., a candidate $H_2^{(-)}$). The naive guess

$$H = \sum_i \sum_{a \neq 3,8} \lambda_i^a \lambda_{i+1}^a \quad (3.2)$$

does not satisfy this criterion. However its truncated version

$$H \equiv H_2^{(+)} = \sum_i \sum_{a \neq 1,2,3,8} \lambda_i^a \lambda_{i+1}^a \quad (3.3)$$

does have the desired property:

$$H_2^{(-)} = \sum_i [\lambda_i^4 \lambda_{i+1}^5 - \lambda_i^5 \lambda_{i+1}^4 + \lambda_i^6 \lambda_{i+1}^7 - \lambda_i^7 \lambda_{i+1}^6] \quad (3.4)$$

³ The existence of an extra mastersymmetry in the XX model is pointed out in [6] but the observation that there are two infinite sequences appears to be new.

⁴ These are normalized as follows:

$$[\lambda^a, \lambda^b] = 2i f^{abc} \lambda^c, \quad \lambda^a \lambda^b + \lambda^b \lambda^a = \frac{4}{3} \delta^{ab} + 2d^{abc} \lambda^c$$

where f^{abc} is completely antisymmetric and d^{abc} completely symmetric.

commutes with $H_2^{(+)}$.⁵ The model (3.3) is thus our candidate XX $su(3)$ model. Note that this particular model has already been identified as a candidate integrable model in [9], picked up as one of the few $su(3)$ models satisfying the Reshetikhin condition [10]. This model also appears as a limiting case of the 19-vertex model found in [11]. In [12] the Reshetikhin condition has been shown to be equivalent to the existence of a boost operator B satisfying $[[B, H], H] = 0$, where if $H = \sum_i h_{i,i+1}$ is the defining hamiltonian, B is given by $B = \sum_i i h_{i,i+1}$. From [9] we thus already know that there exists a mastersymmetry $B_2^{(+)}$, which by itself is a strong signal of integrability: it automatically implies the existence of the conserved charges $H_3^{(+)}$ and $H_4^{(+)}$. Note further that (up to an irrelevant multiplicative factor) $H_2^{(-)}$ can be written as

$$H_2^{(-)} = \sum_i \sum_{a,b \in \mathcal{A}} f^{8ab} \lambda_i^a \lambda_{i+1}^b \quad (3.5)$$

where $\mathcal{A} = \{4, 5, 6, 7\}$. This directly shows that $H_2^{(-)}$ can be obtained from $H_2^{(+)}$ by the application of the mastersymmetry $B_1^{(-)}$ associated to the conserved charge $H_1^{(-)} = \sum_j \lambda_j^8$:

$$[B_1^{(-)}, H_2^{(+)}] = 2iH_2^{(-)} \quad (3.6)$$

(In contrast, the other degree-1 conservation law $\sum \lambda_i^3$ does not lead to a mastersymmetry: $[[\sum_j j \lambda_j^3, H_2^{(+)}], H_2^{(+)}] \neq 0$.)

The existence of the mastersymmetries $B_1^{(-)}$ and $B_2^{(+)}$ singles out the model (3.3). This is the conjectured minimal characterization of an XX model. Its proper characterization, the existence of two infinite families of mastersymmetries, is established in the following section.

⁵ In ref [7] a two-parameter $su(3)$ R matrix is obtained, leading to the spin-chain hamiltonian density

$$\begin{aligned} & \cos \delta (\lambda_i^1 \lambda_{i+1}^1 + \lambda_i^2 \lambda_{i+1}^2 + \lambda_i^6 \lambda_{i+1}^6 + \lambda_i^7 \lambda_{i+1}^7) \\ & - \sin \delta (\lambda_i^1 \lambda_{i+1}^2 - \lambda_i^2 \lambda_{i+1}^1 + \lambda_i^6 \lambda_{i+1}^7 - \lambda_i^7 \lambda_{i+1}^6) \\ & + \cos 2\delta (\lambda_i^4 \lambda_{i+1}^4 + \lambda_i^5 \lambda_{i+1}^5) - \sin 2\delta (\lambda_i^4 \lambda_{i+1}^5 - \lambda_i^5 \lambda_{i+1}^4) \\ & + \cosh \gamma (\lambda_i^3 \lambda_{i+1}^3 + \lambda_i^8 \lambda_{i+1}^8) - \frac{1}{\sqrt{3}} \sinh \gamma (\lambda_i^3 \lambda_{i+1}^8 - \lambda_i^8 \lambda_{i+1}^3) \end{aligned}$$

The first term (proportional to $\cos \delta$) is unitarily equivalent to the above $H_2^{(+)}$ density and the second term is likewise related to $H_2^{(-)}$. However there is no value of the parameters that would make the remaining terms disappear. Our results suggest the existence of a three-parameter deformation of the R matrix of [7] that allows for the decoupling of the undesired terms, leaving us with a one-parameter hamiltonian.

3.2. The $su(3)$ XX conservation laws

To write down the $su(3)$ conservation laws in closed form, we introduce the following notation: Latin indices take values in the set $\mathcal{A} = \{4, 5, 6, 7\}$ and Greek indices are elements of the complementary set $\mathcal{B} = \{1, 2, 3, 8\}$; repeated Latin (Greek) indices are understood to be summed in \mathcal{A} (\mathcal{B}). We have found the general expression for $H_n^{(+)}$ to be

$$H_n^{(+)} = \left(\prod_{i=1}^{n-2} f^{a_i \alpha_i a_{i+1}} \right) \sum_j \lambda_j^{a_1} \lambda_{j+1}^{\alpha_1} \lambda_{j+2}^{\alpha_2} \cdots \lambda_{j+n-2}^{\alpha_{n-2}} \lambda_{j+n-1}^{a_{n-1}} \quad (3.7)$$

while that for $H_n^{(-)}$ reads

$$H_n^{(-)} = f^{8a_0 a_1} \left(\prod_{i=1}^{n-2} f^{a_i \alpha_i a_{i+1}} \right) \sum_j \lambda_j^{a_0} \lambda_{j+1}^{\alpha_1} \lambda_{j+2}^{\alpha_2} \cdots \lambda_{j+n-2}^{\alpha_{n-2}} \lambda_{j+n-1}^{a_{n-1}} \quad (3.8)$$

These expressions are the direct generalization of the $su(2)$ XX conservation laws (i.e., with the replacements $f^{abc} \rightarrow \epsilon^{abc}$, $\mathcal{A} \rightarrow \{1, 2\}$, $\mathcal{B} \rightarrow \{3\}$, they reduce to the $su(2)$ charges $H_n^{(\epsilon)}$ given by (2.1) up to the sign factor $(-1)^{[(n-2)/2] + n(\epsilon+1)/2}$, where $[m]$ indicates the integer part of m). By a direct calculation, we have checked that

$$[H_n^{(+)}, H_2^{(+)}] = [H_n^{(-)}, H_2^{(+)}] = 0 \quad (3.9)$$

and these conservation laws can be recursively generated from $H_2^{(+)}$:

$$\begin{aligned} [B_2^{(+)}, H_n^{(\pm)}] &= (n-1)(H_{n+1}^{(\pm)} \mp 2H_{n-1}^{(\pm)}) \\ [B_1^{(-)}, H_n^{(+)}] &= H_n^{(-)} \end{aligned} \quad (3.10)$$

These verifications use a number of identities:

$$\begin{aligned} f^{a\alpha b} f^{a\alpha c} &= \frac{3}{2} \delta^{bc} \\ f^{8ca} f^{a\alpha b} &= f^{8ba} f^{a\alpha c} \\ f^{ap\alpha} f^{\alpha qb} d^{b\beta a} &= f^{a\beta b} f^{bq\gamma} d^{\gamma pa} \\ f^{\alpha pb} f^{bq\beta} f^{\beta rc} d^{cs\alpha} &= -f^{\alpha qb} f^{bp\beta} f^{\beta sc} d^{cr\alpha} \end{aligned} \quad (3.11)$$

The successive application of the second identity leads to

$$\begin{aligned} &f^{a_1 \alpha_1 a_2} f^{a_2 \alpha_2 a_3} \cdots f^{a_{n-2} \alpha_{n-2} a_{n-1}} f^{8a_{n-1} a_n} \\ &= (-1)^{n-1} f^{a_{n-1} \alpha_{n-2} a_2} f^{a_2 \alpha_{n-3} a_3} \cdots f^{a_{n-2} \alpha_1 a_{n-1}} f^{8a_{n-1} a_1} \end{aligned} \quad (3.12)$$

required in most manipulations involving the $H_n^{(-)}$'s. In this last identity there are $n - 1$ factors f and it should be noticed that the free indices $a_1, \alpha_1, \dots, \alpha_{n-2}, a_n$ on the left hand side are read in the reverse order on the right hand side.

These identities also ensure the existence of the mastersymmetries $B_n^{(\pm)}$ whose commutation relations with the conservation laws have exactly the structure (2.6). In particular, we have

$$[B_n^{(\pm)}, H_2^{(+)}] = H_{n+1}^{(\pm)} + 2H_{n-1}^{(\pm)} \quad (3.13)$$

where as in the $su(2)$ case, if $H_n^{(\pm)} = \sum_j [h_n^{(\pm)}]_{j j+1 \dots j+n-1}$, then $B_n^{(\pm)} = \sum_j j [h_n^{(\pm)}]_{j j+1 \dots j+n-1}$. These relations suffice to demonstrate the presence of two infinite family of mastersymmetries.

The comparison of the $su(3)$ conservation laws with those of $su(2)$ suggests the existence of an analogue of the Jordan-Wigner transformation. However such a transformation does not have the expected locality property. No generalized Jordan-Wigner transformation has been actually found.

3.3. The $su(3)$ XX R matrix

If instead of Gell-Mann matrices, we use the matrices $E^{\alpha\beta}$ with zeros everywhere except for a 1 at the intersection of line α and column β , the hamiltonian reads

$$H_2^{(\pm)} = \sum_i \sum_{\alpha=1,2} [E_i^{3\alpha} E_{i+1}^{\alpha 3} \pm E_i^{\alpha 3} E_{i+1}^{3\alpha}] \quad (3.14)$$

Note that this hamiltonian density is unitarily equivalent to

$$\sum_i \sum_{\alpha=1,2} [E_i^{\alpha \alpha+1} E_{i+1}^{\alpha+1 \alpha} \pm E_i^{\alpha+1 \alpha} E_{i+1}^{\alpha \alpha+1}] \quad (3.15)$$

where only the matrices corresponding to the generator associated to the simple roots are seen to appear. The unitary transformation matrix relating the two hamiltonians is built out of the matrix

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (3.16)$$

which permutes the vectors 2 and 3 while leaving the vector 1 invariant. The unitary operator $U \otimes \dots \otimes U$, one copy for each site of the chain, transforms one hamiltonian into the other. Such vector permutations also mean that the whole construction still goes through for a hamiltonian $H_2^{(+)}$ (see (3.3)) defined with $a \neq 6, 7, 3, 8$, or $a \neq 4, 5, 3, 8$.

The R matrix leading to (3.14) is

$$\begin{aligned}\check{R}(\lambda) = & \sin \lambda \sum_{\alpha=1,2} (e^{i\delta} E^{\alpha 3} \otimes E^{3\alpha} + e^{-i\delta} E^{3\alpha} \otimes E^{\alpha 3}) \\ & + \sum_{\alpha=1,2} (E^{\alpha\alpha} \otimes E^{33} + E^{33} \otimes E^{\alpha\alpha}) + \cos \lambda [E^{33} \otimes E^{33} + \sum_{\alpha,\beta=1,2} E^{\alpha\alpha} \otimes E^{\beta\beta}]\end{aligned}\quad (3.17)$$

where $\check{R} = PR$, P being the permutation operator. (Note that $\check{R}(0) = \text{Id.}$) This is related to our $su(3)$ model in the sense that

$$\frac{d\check{R}}{d\lambda}(0) = \sum_{\alpha=1,2} [(E^{\alpha 3} \otimes E^{3\alpha} + E^{3\alpha} \otimes E^{\alpha 3}) \cos \delta + i(E^{\alpha 3} \otimes E^{3\alpha} - E^{3\alpha} \otimes E^{\alpha 3}) \sin \delta]\quad (3.18)$$

Evaluated at sites $i, i+1$, this last expression is a linear combination of the densities of $H_2^{(\pm)}$. That \check{R} is a genuine R matrix follows from the fact that it satisfies the Yang-Baxter equation:

$$\check{R}_{12}(\lambda - \mu) \check{R}_{23}(\lambda) \check{R}_{12}(\mu) = \check{R}_{23}(\mu) \check{R}_{12}(\lambda) \check{R}_{23}(\lambda - \mu) \quad (3.19)$$

This provides, in the framework of the quantum inverse scattering method (QISM), an independent proof of integrability. And as in the $su(2)$ case, the presence of the residual parameter δ explains the origin of the two sequences of conservation laws found for this model.

We now show how the $su(2)$ and $su(3)$ models and their features generalize straightforwardly to $su(N)$.

4. The $su(N)$ generalization

Having unraveled the Lie algebraic structure of the $su(3)$ XX model in the previous section, the generalization to the $su(N)$ case is clear. Writing $\alpha < 3$ instead of $\alpha = 1, 2$, the generalization simply amounts to replacing 3 by N in the previous formulae. The $su(N)$ XX hamiltonians, for $N \geq 2$, are

$$H_2^{(\pm)} = \sum_i \sum_{\alpha < N} [E_i^{\alpha N} E_{i+1}^{N\alpha} \pm E_i^{N\alpha} E_{i+1}^{\alpha N}] \quad (4.1)$$

and the R matrix reads

$$\begin{aligned}\check{R}(\lambda) = & \sum_{\alpha < N} (xE^{\alpha N} \otimes E^{N\alpha} + x^{-1}E^{N\alpha} \otimes E^{\alpha N}) \sin \lambda \\ & + \sum_{\alpha < N} (E^{\alpha\alpha} \otimes E^{NN} + E^{NN} \otimes E^{\alpha\alpha}) \\ & + [E^{NN} \otimes E^{NN} + \sum_{\alpha, \beta < N} E^{\alpha\alpha} \otimes E^{\beta\beta}] \cos \lambda\end{aligned}\tag{4.2}$$

where $x = e^{i\delta}$. It satisfies the regularity property

$$\check{R}(0) = \text{Id}\tag{4.3}$$

the unitarity condition

$$\check{R}(\lambda)\check{R}(-\lambda) = \text{Id} \cos^2 \lambda\tag{4.4}$$

and the Yang-Baxter equation (3.19) (and as already mentioned, this R matrix appears to be new). In the QISM formulation, the conserved quantities can be obtained from the transfer matrix

$$\tau(\lambda) \equiv \text{tr}_0 T(\lambda) \equiv \text{tr}_0 L_{0M} \dots L_{01}\tag{4.5}$$

(for a chain of M sites) where $L_{0i}(\lambda) = R_{0i}(\lambda)$, the trace is taken over the auxiliary space 0 and $T(\lambda)$ is the monodromy matrix. The two-dimensional vertex model has $N^2 + 2N - 2$ non-vanishing Boltzmann weights. The Yang-Baxter equation ensures that two transfer matrices with different spectral parameters commute:

$$[\tau(\lambda), \tau(\mu)] = 0\tag{4.6}$$

Therefore $\tau(\lambda)$ is a generating function for the conserved quantities of the one-dimensional spin-chain model. The usual choice

$$H_{n+1} \propto \left(\frac{d^n \ln \tau(\lambda)}{d\lambda^n} \right)_{\lambda=0}, \quad n \geq 0\tag{4.7}$$

gives the mutually commuting conserved quantities studied above. For instance we find

$$\begin{aligned}H_2 &= H_2^{(+)} \cos \delta + iH_2^{(-)} \sin \delta \\ H_3 &= H_3^{(+)} \cos(2\delta) + iH_3^{(-)} \sin(2\delta)\end{aligned}\tag{4.8}$$

where

$$H_3^{(\pm)} = \sum_i \left[\sum_{\alpha < N} (E_i^{\alpha N} E_{i+1}^{NN} E_{i+2}^{N\alpha} \mp E_i^{N\alpha} E_{i+1}^{NN} E_{i+2}^{\alpha N}) \right. \\ \left. + \sum_{\alpha, \beta < N} \left(\pm E_i^{N\alpha} E_{i+1}^{\alpha\beta} E_{i+2}^{\beta N} - E_i^{\alpha N} E_{i+1}^{\beta\alpha} E_{i+2}^{N\beta} \right) \right] \quad (4.9)$$

Since H_2 and H_3 commute for all values of δ we find that the four hamiltonians H_2^\pm, H_3^\pm mutually commute, as expected.

In addition to these conserved charges, the transfer matrix formalism makes quite transparent the fact that there are additional spin-1 conserved charges:

$$H_1^{\alpha\beta} = \sum_j E_j^{\alpha\beta} \quad \alpha, \beta < N \\ H_1^\alpha = \sum_j (E_j^{\alpha\alpha} - E_j^{NN}) \quad \alpha < N \quad (4.10)$$

To prove the commutativity of these charges with the transfer matrix, we observe that

$$[E_i^{\alpha\beta}, L_{0i}(\lambda)] = -[E_0^{\alpha\beta}, L_{0i}(\lambda)] \quad (4.11)$$

The commutator $[H_1^{\alpha\beta}, \tau(\lambda)]$ is then easily seen to vanish:

$$[H_1^{\alpha\beta}, \tau(\lambda)] = \text{tr}_0 \sum_{i=1}^M (L_{0M} \cdots L_{0i+1} [E_i^{\alpha\beta}, L_{0i}(\lambda)] L_{0i-1} \cdots L_{01}) \\ = -\text{tr}_0 \sum_{i=1}^M (L_{0M} \cdots L_{0i+1} [E_0^{\alpha\beta}, L_{0i}(\lambda)] L_{0i-1} \cdots L_{01}) \\ = -\text{tr}_0 (E_0^{\alpha\beta} L_{0M} \cdots L_{01}) + \text{tr}_0 (L_{0M} \cdots L_{01} E_0^{\alpha\beta}) = 0 \quad (4.12)$$

A similar argument works for H_1^α . To the defining hamiltonian of the XX model, we can thus add $N^2 - 2N + 1$ independent magnetic-field terms without spoiling the integrability of the model. The spin-1 charges can also be seen as the generators of an $su(N-1) \oplus u(1)$ symmetry.

In the generic $su(N)$ case, the conservation laws still take the form (3.7) and (3.8) with the following redefinition of the sets \mathcal{A} and \mathcal{B} : the elements of \mathcal{A} are those corresponding to the $su(N)$ generators constructed from $E^{\alpha N}$ and $E^{N\alpha}$, e.g.,

$$\lambda^{2j} = E^{jN} + E^{Nj}, \quad \lambda^{2j+1} = -i(E^{jN} - E^{Nj}) \quad 1 \leq j < N \quad (4.13)$$

while those of \mathcal{B} correspond to the generators constructed from E^{NN} and $E^{\alpha\beta}$, $(\alpha, \beta < N)$. The matrix λ^8 is replaced by $\lambda^{N^2-1} = \sum_{j < N} E^{jj} - (N-1)E^{NN}$. Furthermore, the mastersymmetries obtained for the $su(2)$ and $su(3)$ models generalize directly to the $su(N)$ case.

We stress again that the special role played by the generators corresponding to the index N is only superficial. All summations encountered above with a restriction ‘ α less than N ’ can be replaced with the restriction ‘ α different from α_0 ’ with any fixed value of α_0 between 1 and N . The models defined for different values α_0 are all related to each other by unitary transformations which permute the basis vectors (cf. the relation between the hamiltonians (3.14) and (3.15)).

With this equivalence relation in mind, it is now instructive to compare the XX hamiltonians with their XXX and XXZ counterparts for $su(N)$. These hamiltonians can be written as a sum of two pieces:

$$H_2 = H_2^\Delta + H_2^{CSA} = \sum_i \sum_{\alpha \in \Delta^+} (\lambda_i^\alpha \lambda_{i+1}^{-\alpha} + \lambda_i^{-\alpha} \lambda_{i+1}^\alpha) + H_2^{CSA} \quad (4.14)$$

where Δ^+ is the set of *positive* roots of the $su(N)$ algebra and H_2^{CSA} is bilinear in the generators of Cartan subalgebra. The XXZ deformation of XXX corresponds to a modification of the H_2^{CSA} part of the XXX hamiltonian. The XX hamiltonian in contrast has no CSA contribution *and* the sum over the raising and lowering operators is restricted to a subset of $N-1$ roots.

We now diagonalize the set of conserved quantities found earlier.

5. Algebraic Bethe Ansatz

Since all the hamiltonians $H_n^{(\pm)}$ mutually commute it is possible to simultaneously diagonalize them. The R matrix formulation provides a powerful and elegant diagonalization procedure of the transfer matrix $\tau(\lambda)$ and therefore of all the hamiltonians. This method is known as the algebraic Bethe Ansatz (for a review see [13]).

The transfer matrix was defined in (4.5) as the trace over the auxiliary space of the monodromy matrix $T(\lambda)$. The latter is an N -dimensional matrix whose entries are operators acting on the Hilbert space $C^N \otimes \dots \otimes C^N$, with a copy for every site. Some elements of the monodromy matrix are used to create Ansätze for the eigenstates. The Yang-Baxter equation (3.19) implies the RTT-relation:

$$\check{R}(\lambda - \mu) T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda) \check{R}(\lambda - \mu) \quad (5.1)$$

Written in components this equation provides the algebraic relations needed to find the action of the transfer matrix on the states.

We now use the following notation for the monodromy matrix:

$$T = \begin{pmatrix} t_{11} & \cdot & \cdot & \cdot & t_{1,N-1} & B_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ t_{N-1,1} & \cdot & \cdot & \cdot & t_{N-1,N-1} & B_{N-1} \\ C_1 & \cdot & \cdot & \cdot & C_{N-1} & S \end{pmatrix} \quad (5.2)$$

With this notation the transfer matrix becomes $\tau(\lambda) = S(\lambda) + \sum_{a=1}^{N-1} t_{aa}(\lambda)$.

It is easy to see that the vector $||N\rangle \equiv |N\dots N\rangle$ is an eigenvector of the transfer matrix. The action of the monodromy matrix on the vector $||N\rangle$ is given by

$$T(\lambda) ||N\rangle = \begin{pmatrix} t(\lambda) & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & t(\lambda) & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & t(\lambda) & 0 \\ C_1 & C_2 & \cdot & \cdot & \cdot & C_{N-1} & s(\lambda) \end{pmatrix} ||N\rangle \quad (5.3)$$

where $t(\lambda) = a(\lambda)^M$, $s(\lambda) = b(\lambda)^M$, and $a(\lambda) = x^{-1} \sin \lambda$, $b(\lambda) = \cos \lambda$. We therefore take as Ansatz eigenvectors the following linear combination of states:

$$|\lambda_1, \dots, \lambda_p\rangle = \sum_{a_p, \dots, a_1=1}^{N-1} F^{a_p, \dots, a_1} C_{a_1}(\lambda_1) \dots C_{a_p}(\lambda_p) ||N\rangle \quad (5.4)$$

where $p \leq M$. The parameters λ_i and the coefficients F are to be determined.

The equation (5.1) gives the following relations:

$$\begin{aligned} C_a(\lambda)C_b(\mu) &= C_a(\mu)C_b(\lambda) = C_c(\mu)C_d(\lambda)P_{dc,ab} \\ S(\lambda)C_a(\mu) &= f(\mu - \lambda)C_a(\mu)S(\lambda) + g(\mu - \lambda)C_a(\lambda)S(\mu) \\ t_{ab}(\lambda)C_c(\mu) &= f(\lambda - \mu)C_d(\mu)t_{ae}(\lambda)P_{ed,bc} + g(\lambda - \mu)C_b(\lambda)t_{ac}(\mu) \\ f(\lambda) &= \frac{x \cos \lambda}{\sin \lambda}, \quad g(\lambda) = -\frac{x}{\sin \lambda}, \end{aligned} \quad (5.5)$$

where all the indices belong to $\{1, \dots, N-1\}$ and P is now the permutation operator for two copies of the $(N-1)$ -dimensional space.

One then applies the transfer matrix on the state $|\lambda_1, \dots, \lambda_p\rangle$ and with the help of the above relations commutes the t_{aa} and S through the C_i 's. This creates two types of terms:

the wanted terms which are proportional to the original eigenvector, and the unwanted terms which are required to vanish. The S and $\sum_a t_{aa}$ contributions to the wanted terms are respectively

$$\begin{aligned} s(\lambda) \prod_{j=1}^p f(\lambda_j - \lambda) |\lambda_1, \dots, \lambda_p\rangle \\ t(\lambda) \prod_{j=1}^p f(\lambda - \lambda_j) C(\lambda_1) \otimes \dots \otimes C(\lambda_p) \tau^{(N-1,p)} F ||N\rangle \end{aligned} \quad (5.6)$$

where the matrix $\tau^{(N-1,p)}$ is a transfer matrix for a p -sites chain with $N - 1$ states on each site. In order for the t contribution to give a vector proportional to $|\lambda_1, \dots, \lambda_p\rangle$ one requires F to be an eigenvector of $\tau^{(N-1,p)}$, with eigenvalue $\Lambda^{(N-1,p)}$.

The unwanted terms arising from S and t have a structure similar to the wanted terms but with one parameter λ_j replaced with λ . Requiring these contributions to vanish one finds

$$\tau^{(N-1,p)} F = (-1)^{p-1} \left(\frac{b(\lambda_j)}{a(\lambda_j)} \right)^M F, \quad j = 1, \dots, p \quad (5.7)$$

The resulting eigenvalue of the XX model is:

$$\Lambda^{(N,M)}(\lambda) = s(\lambda) \prod_{j=1}^p f(\lambda_j - \lambda) + t(\lambda) \left(\prod_{j=1}^p f(\lambda - \lambda_j) \right) \Lambda^{(N-1,p)} \quad (5.8)$$

One then has to diagonalize the transfer matrix $\tau^{(N-1,p)}$. The crucial points here are that this transfer matrix is built out of the permutation operator appearing in the commutation relations, and that the latter operator is in turn an R matrix. Thus one can, in principle, embed the matrix $\tau^{(N-1,p)}$ in an $su(N - 1)$ XXX or XXZ model and apply the foregoing procedure repeatedly until the dimension of the lower level space becomes equal to one; at this point one has $\tau^{(1,p')} = 1$ and the nested Ansatz closes. Note that for $N = 2$ no nesting is necessary.

However, a simplification to this procedure occurs. The key point is that the transfer matrix $\tau' \equiv \tau^{(N-1,p)}$ is not only constant, but also the *unit-shift* operator on a periodic p -sites chain, with $N - 1$ possible states on each site. Therefore τ' can be written as a product of disjoint permutation cycles; moreover one has $(\tau')^p = \text{Id}$. The eigenvalues $\Lambda^{(N-1,p)}$ are then *at most* p^{th} roots of unity and are highly degenerate. The dimensions of the cycles and their multiplicities will depend on both p and N . The coefficients F are the

eigenvectors of τ' . For every eigenvalue $\Lambda^{(N-1,p)}$ one gets an eigenvalue $\Lambda^{(N,M)}(\lambda)$ from equation (5.8), where the parameters λ_j are solutions of the closing equations (see (5.7)):

$$(-1)^{p-1} \left(\frac{b(\lambda_j)}{a(\lambda_j)} \right)^M = \Lambda^{(N-1,p)} \quad , \quad j = 1, \dots, p \quad (5.9)$$

This form is the generalization of the $su(2)$ BA equations, for which $\Lambda^{(N-1,p)} = 1$!

The vectors $||i\rangle$, $i = 1, \dots, N-1$, are also eigenvectors of the XX transfer matrix. Some Bethe Ansatz states are built using only one type of creation operators (for a given i), and the resulting Bethe Ansatz equations are $su(2)$ equations.

We have thus completely diagonalized the transfer matrix of the XX $su(N)$ model. The nested Bethe Ansatz truncates and the spectrum is trivial in the above sense. This is one more indication that our generalized XX model is the natural generalization of the $su(2)$ model.

We now make some remarks. The appearance of the permutation operator for the $su(N-1)$ model was to be expected since building the eigenstates over the vector $||N\rangle$ effectively removes the dimension corresponding to the basis vector $|N\rangle$. The \check{R} matrix (4.2) is then clearly seen to yield $R^{(N-1,N-1)} = P$ when all terms involving N are dropped.

Finally, for $N > 2$, we show that the eigenvectors $|\lambda_1, \dots, \lambda_p\rangle$ are generically not eigenvectors of the magnetic field operators (4.10) studied in section 4. Because the spectrum is degenerate this is not in contradiction with the fact that the magnetic operators commute with the conserved quantities. Using the same methods as in section 4 we derive the following relations:

$$\begin{aligned} [H_1^\alpha, C_i(\lambda)] &= (1 + \delta_{i,\alpha}) C_i(\lambda) \quad , \quad i, \alpha < N \\ [H_1^{\alpha\beta}, C_i(\lambda)] &= \delta_{i,\beta} C_\alpha(\lambda) \quad , \quad i, \alpha, \beta < N \\ \left[\sum_{j=1}^M E_j^{NN}, C_i(\lambda) \right] &= -C_i(\lambda) \quad , \quad i, \alpha < N \end{aligned} \quad (5.10)$$

Therefore the Bethe states are eigenvectors only for one magnetic operator, with

$$\sum_{j=1}^M E_j^{NN} |\lambda_1, \dots, \lambda_p\rangle = (M - p) |\lambda_1, \dots, \lambda_p\rangle . \quad (5.11)$$

One then shows that the Ansatz (5.4) vanishes identically if $p > M$.

6. Conclusion

We have presented the natural $su(N)$ generalization of the XX model and proved its complete integrability both directly and via the quantum inverse scattering method. The hamiltonians and the other conserved charges were seen to fit in a unifying pattern. The Bethe Ansatz diagonalization of the transfer matrix showed that the spectrum is trivial in a certain sense. This is another indication that the XX $su(N)$ models we introduced are the natural generalization of the $su(2)$ model

Note that the projection of these $su(N)$ XX models onto $su(2)$ leads to integrable higher spin (i.e., $j = (N - 1)/2$) versions of the $su(2)$ XX model, with non-quadratic defining hamiltonians.

There are obvious extensions to this work. One is to consider the generalization to other Lie algebras. Also, it would be interesting to find the multiparametric quantum-group symmetry underlying the $su(N)$ XX model. We expect that there exists a multiparametric hamiltonian interpolating continuously between the XX and XXZ model, as happens for $su(2)$.

An interesting application of the present analysis is the construction of integrable generalizations of the Hubbard model. This model is known to be integrable in one dimension [2]; its integrable structure is different from the usual spin chains models and the known $su(2)$ model stands alone outside an integrable hierarchy. The $su(N)$ hamiltonian generalizing the Hubbard model is to be built from two independent copies of the $su(N)$ XX model that couple through their $H_1^{(-)}$ densities at the same site. As mentioned in the introduction, a further support for the correctness of our characterizing properties of the XX model is rooted in the integrability of this generalized Hubbard model. A strong integrability indicator is the existence of a spin-3 conserved charge that couples the two independent XX models along the $H_2^{(-)}$ density of one model to the $H_1^{(-)}$ density of the other, exactly as in the $su(2)$ case [6]. The details of the analysis of this $su(N)$ Hubbard model will be presented elsewhere.

Acknowledgment

We thank Yvan Saint-Aubin for a useful discussion.

References

1. M. Jimbo, Comm. Math. Phys. **102** (1986) 537; V.V. Bazhanov, Comm. Math. Phys. **113** (1987) 471.
2. B.S. Shastry, Phys. Rev. Lett. **56** (1986) 1529, 2453.
3. J. Perk and C.L. Schultz, Phys. Lett. **84A** (1981) 407; A. Foerster, I. Roditi and L. Rodrigues, Mod. Phys. Lett. **A 11** (1996) 987.
4. E.V. Gusev, Theor. Math. Phys., **53** (1983) 1018; M. Grady, Phys. Rev **D25** (1982) 1103; H. Itoyama and H. B. Thacker, Nucl. Phys. **B320** (1989) 541.
5. H. Araki, Comm. Math. Phys., **132** (1990) 155.
6. M. P. Grabowski and P. Mathieu, Ann. Phys. **243** (1995) 299.
7. J. Abad and M. Rios, J. Phys. A **28** (1995) 3319.
8. E. Barouch and B. Fuchssteiner, Stud. Appl. Math. **73** (1985) 221.
9. K.-H. Mütter and A. Schmitt, J. Phys. **A 28** (1995) 2265.
10. Kulish P P and Sklyanin E K 1982 in *Integrable Quantum Field Theories* ed J Hietarinta and C Montonen (Lecture Notes in Physics **151**, Springer Verlag) p. 61.
11. M. Idzumi, T. Tokihiro and M. Arai, J. de Physique **I4** (1994) 1151.
12. M. P. Grabowski and P. Mathieu, J. Phys. **A 28** (1995) 4777.
13. H.J. De Vega, Int. J. Mod. Phys. **A 4** (1989) 2371–2463, and references therein.